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Coning during withdrawal from two fluids of different density in a porous medium

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Abstract The steady response of the interface between two fluids with different density in a porous medium is considered during extraction through a line sink. Supercritical withdrawal, or coning as it is often called, in which both fluids are being withdrawn, is investigated using a coupled integral equation formulation. It is shown that for each entry angle of the interface into the sink there is a range of supercritical solutions that depend on the flow rate, and that as the flow rate decreases the cone narrows. As the magnitude of the entry angle increases this range of flow-rate values decreases to a narrow range as the entry becomes vertical. Only one branch of solutions (that with horizontal entry) has the property that the interface levels off at a finite height, and this is investigated as a separate branch of solution.

Keywords Boundary-integral method · Critical withdrawal · Line sink · Porous media · Supercrital withdrawal

1 Introduction

Extraction of fluids from within porous media is of importance in groundwater aquifers and oil reservoirs among many other applications. Usually, oil lies above water and below gas, and fresh water often lies above salt water. When fluid is withdrawn in this situation, the fluid will come from the fluid layer surrounding the point of removal unless the pumping rate is high enough to pull the interface directly into the outlet, a phenomenon known as coning. If the withdrawal rate is constant below this critical rate and the pressure forces acting on the fluid are in equilibrium with the gravity force, the interface will reach a stable shape below the well. The critical flow rate is defined as the maximum rate at which only the fluid adjacent to the sink is withdrawn. At a higher supercritical rate, some of both fluids will be removed.

In mathematical analysis the interface between the two fluids is often assumed to be sharp where the two fluids do not mix. The determination of the critical withdrawal rate is of great practical interest. Intensive study has been carried out since the work of Muskat and Wyckoff [1], and many scientists have studied critical withdrawal by

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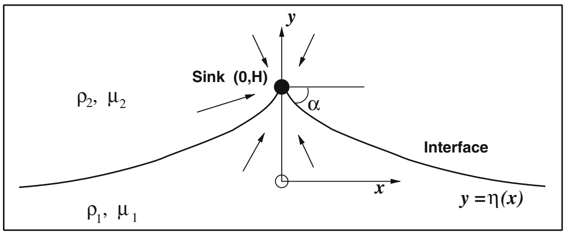


Fig. 1 Schematic of the flow into a link sink from two layers of different density—supercritical flow rate

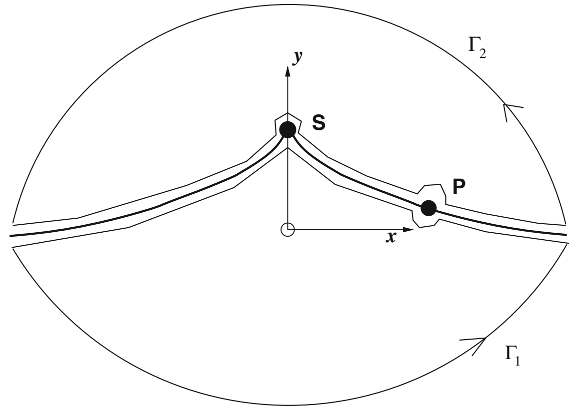


Fig. 2 Contours used in formulation of the integral equation. The contours avoid the sink S and the point z_0, denoted P. The contribution to the integral from the two radial arcs as $r \to \infty$ is zero, leaving just the integral along the interface

using analytical methods for various aquifer configurations; see for example [2–6]. In particular, Bear and Dagan [2] computed the critical, single-fluid flow in an unbounded medium.

The analogous problem of supercritical withdrawal in two-layer surface water bodies was considered in [7–9] where an integral-equation approach is used to compute accurate numerical solutions. However, limited research has been done for supercritical coning flows in porous media. Yu et al. [10] and Henderson et al. [11] used a finite-difference method to simulate an isothermal, monophasic, highly compressible flow in a supercritical condition.

In the present study, two fluids of different density separated by an interface of infinitesimal thickness within an homogeneous and isotropic aquifer in two dimensions are considered. A line sink (a point in two-dimensions) is located in the upper fluid and withdraws at some constant rate (see Fig. 1). We seek coning solutions in which both fluids are flowing out through the sink. Integral equations to be satisfied in both fluids and equations matching the pressure across the interface are derived and solved numerically. A study of the effect of variations in several parameters is conducted. It is found that for each value of entry angle of the interface into the sink there are multiple solutions over a range of flow values. In each case, as the flow rate decreases, the interface near to (but not at) the sink steepens until it becomes vertical, at which point the method fails. This range of values narrows as the magnitude of the entry angle increases. In all cases, except that for which the entry is horizontal, the interface levels off at infinite values of elevation due to the logarithmic nature of the sink flow and the interface condition (as it does for single-fluid, subcritical flows). If the interface enters the sink horizontally, there is a separate branch of solution in which the surface levels off at a finite elevation.

2 Formulation

2.1 Equations

Consider an homogeneous and isotropic porous medium with intrinsic permeability κ. Two fluids of different density and dynamic viscosity are separated by an interface of infinitesimal thickness as seen in Fig. 1. A line sink (S) is located at a distance, H, above the origin. The sink extracts a total volume per unit time per unit width of Q. The fluids located beneath and above the impermeable boundary are defined as fluids 1 and 2, respectively.

Using complex variables, we let the physical plane correspond to the z-plane shown in Fig. 1 where $z = x + iy$. The origin is located a distance H directly below the sink. If $y = \eta(x)$ is the equation of the interface, suppose the fluid below the interface to have density ρ_1 and viscosity μ_1 and the fluid above the interface to have density ρ_2 and viscosity μ_2. We define a potential function in each of the fluids as

$$\begin{aligned} \Phi_1 &= \frac{\kappa}{\mu_1}(p+\rho_1 g y), \quad y<\eta(x), \\ \Phi_2 &= \frac{\kappa}{\mu_2}(p+\rho_2 g y), \quad y>\eta(x), \end{aligned} \tag{1}$$

where p is the pressure at elevation y, g is gravitational acceleration and κ is the intrinsic permeability of the medium. Therefore, the velocity or specific discharge is given by

$$\mathbf{q}_j = -\nabla\Phi_j, \quad j=1,2. \tag{2}$$

Matching the pressure across the interface between the two immiscible fluid regions gives the condition on the interface, $y=\eta(x)$, that

$$\Phi_1 - \gamma\Phi_2 = Ky, \quad \text{where } \gamma = \frac{\mu_2}{\mu_1} \quad \text{and} \quad K = \frac{\kappa g(\rho_1-\rho_2)}{\mu_1}. \tag{3}$$

When the withdrawal rate is below critical, the lower fluid is assumed to be motionless and hence to be at a constant potential. It is noted that, since the potential due to the sink is logarithmic, then if only one fluid is flowing the condition on the interface (3) leads to an interface of unbounded elevation as x approaches infinity. This situation carries over into the supercritical case as well, and so, in general, the interface does not level off at a finite elevation. It is tempting to think that this may be due to the geometry chosen, e.g no impermeable base above (or beneath) the sink, but this geometry still gives the same logarithmic behaviour.

However, there is a special case in which the potentials in the two regions can cancel each other exactly if the mass flux from each of the two fluids matches, i.e., if the interface enters the sink horizontally. In that case, it is possible that the interface becomes horizontal at a finite value of elevation as x becomes large.

2.2 Boundary-integral method for supercritical withdrawal

The solutions we seek are those in which the interface is drawn up a distance H to a point where it enters the sink with an angle α to the horizontal, as shown in Fig. 1. Since flux from each fluid (see below) depends on the angle of entry, α, then from the right half-plane the flow volume per time per unit width from the lower fluid is $Q(\frac{\pi}{2}-\alpha)/\pi$ and from the upper fluid it is $Q(\frac{\pi}{2}+\alpha)/\pi$. Fluid is withdrawn from both above and below the interface. The velocity potentials of the separate flow fields below and above the interface must satisfy Laplace's equation,

$$\begin{aligned} \nabla^2\Phi_1(x,y) &= 0, \quad y<\eta(x), \\ \nabla^2\Phi_2(x,y) &= 0, \quad y>\eta(x). \end{aligned} \tag{4}$$

In addition, since we are dealing with potential theory, it is possible to define a stream function, $\Psi_j(x,y)$, $j=1,2$ for each fluid such that

$$\begin{aligned} \nabla^2\Psi_1(x,y) &= 0, \quad y<\eta(x), \\ \nabla^2\Psi_2(x,y) &= 0, \quad y>\eta(x), \end{aligned} \tag{5}$$

and streamlines of the flow correspond to lines of constant value of Ψ_j, $j=1,2$. The condition that there be no flow across the interface can then be satisfied by enforcing the condition that the stream functions are constant along the interface, i.e.,

$$\Psi_1 = \Psi_2 = 0 \quad \text{on } y=\eta(x). \tag{6}$$

As the sink is approached, the velocity potentials must have the correct behaviour, which is

$$\begin{aligned} \Phi_1 &\to \frac{Q_1}{\frac{\pi}{2}-\alpha}\log(x^2+(y-H)^2)^{1/2} \quad \text{as } (x,y)\to(0,H),\ y<\eta(x), \\ \Phi_2 &\to \frac{Q_2}{\frac{\pi}{2}+\alpha}\log(x^2+(y-H)^2)^{1/2} \quad \text{as } (x,y)\to(0,H),\ y>\eta(x), \end{aligned} \tag{7}$$

where Q_1 and Q_2 are the respective total dimensional fluxes per unit width (from the right half-plane) from within the two regions. There is a relationship between these two values which must hold if the dynamic condition on the interface is to be satisfied. Applying Darcy's Law [12] to the streamline along the interface, and noting that for steady flow there must be no pressure difference across the interface we have

$$\eta(x) = \frac{1}{K}\left[\Phi_1(x, y) - \gamma\Phi_2(x, y)\right]. \tag{8}$$

Considering the behaviour of the flow near the sink (7) and differentiating the interface condition (8) with respect to arclength, s, it follows that

$$q_1 - \gamma q_2 = K\frac{\mathrm{d}\eta(x)}{\mathrm{d}s} = K\sin\alpha \tag{9}$$

where q_1 and q_2 are Darcy velocities in the two fluids. If the flow into the sink is radial, we have

$$\frac{\gamma Q_2}{2r_d\left(\frac{\pi}{2}+\alpha\right)} - \frac{\gamma Q_1}{2r_d\left(\frac{\pi}{2}-\alpha\right)} = K\sin\alpha, \tag{10}$$

where r_d is the radius of the pump. As $r_d \to 0$, it follows that

$$\frac{Q_2}{Q_1} = \frac{\left(\frac{\pi}{2}+\alpha\right)}{\left(\frac{\pi}{2}-\alpha\right)}. \tag{11}$$

The total flux per unit width is $Q = Q_1 + Q_2$ and the flux from the two layers only matches if $\alpha = 0$.

Defining the following dimensionless variables,

$$y^* = y/H, \quad x^* = x/H, \quad \Phi_1^* = \Phi_1\Big/\frac{Q_1}{\left(\frac{\pi}{2}-\alpha\right)} \quad \text{and} \quad \Phi_2^* = \Phi_2\Big/\frac{\gamma Q_2}{\left(\frac{\pi}{2}+\alpha\right)},$$

we derive the non-dimensional form of the dynamic interface condition (8):

$$\eta^* = \frac{2\gamma\pi}{\pi(1+\gamma)+2\alpha(1-\gamma)}G(\Phi_1^* - \Phi_2^*) \quad \text{where } G = \frac{Q}{\pi K H} \tag{12}$$

and

$$\begin{aligned}
\Phi_1^* &\to \log(x^{*2} + (y^*-1)^2)^{1/2} \quad \text{as } (x^*, y^*) \to (0, 1), \quad y^* < \eta^*(x^*),\\
\Phi_2^* &\to \log(x^{*2} + (y^*-1)^2)^{1/2} \quad \text{as } (x^*, y^*) \to (0, 1), \quad y^* > \eta^*(x^*).
\end{aligned} \tag{13}$$

The asterisk denotes dimensionless variables and will be dropped for simplicity. The choice of H as a length scale is slightly unusual since, in general, there is no actual length scale in this steady problem except that obtained as a combination of the other variables. However, in this problem we can think of H as being the initial elevation of the sink above the interface when it is at rest. This choice is consistent with the flows described at the end of Sect. 3, where the interface enters the sink horizontally and levels off at a finite distance beneath the sink.

The quantity G is a measure of the flow strength and perhaps the most important parameter in the problem.

Using complex variable theory we construct a complex potential for each fluid that consists of the potential and the stream function and builds in the correct behaviour both near the sink and in the far field. The goal is then to compute the corrections to these that satisfy the equations. Options which satisfy these requirements are;

$$\begin{aligned}
f_1 &= \Phi_1 + \mathrm{i}\Psi_1 = \log(z-\mathrm{i}) - \frac{2\alpha}{\pi}\log\left(z - \mathrm{i}\frac{\pi}{2\alpha}\right) + w_1, \quad y < \eta(x),\\
f_2 &= \Phi_2 + \mathrm{i}\Psi_2 = \log(z-\mathrm{i}) + \frac{2\alpha}{\pi}\log\left(z + \mathrm{i}\frac{\pi}{2\alpha}\right) + w_2, \quad y > \eta(x),
\end{aligned} \tag{14}$$

where α is the angle of the interface at the point of entry into the sink and $w_j = \phi_j + \mathrm{i}\psi_j$, $j = 1, 2$, are the correction terms for the full complex potentials. In each fluid, this form represents the line sink at $(x, y) = (0, 1)$ and the addition of another singular point outside the domain of interest; a line sink at $x = 0$, $y = \frac{\pi}{2\alpha}$ for the lower fluid and a line source at $x = 0$, $y = -\frac{\pi}{2\alpha}$ for the upper fluid. These choices satisfy the requirement that the line given by $\Psi_j = 0$, $j = 1, 2$ enters the sink at an angle α to the horizontal, provided

$$\begin{aligned}
\psi_1(x,\eta) &= -\arctan\left(\frac{\eta(x)-1}{x}\right) - \frac{2\alpha}{\pi}\arctan\left(\frac{\eta(x)-\pi/2\alpha}{x}\right),\\
\psi_2(x,\eta) &= -\arctan\left(\frac{\eta(x)-1}{x}\right) + \frac{2\alpha}{\pi}\arctan\left(\frac{\eta(x)+\pi/2\alpha}{x}\right).
\end{aligned}\tag{15}$$

The choice of f_1 and f_2 also ensures that $w_j \to 0$, $j = 1, 2$ as $|z| \to \infty$ or as $z \to \mathrm{i}$. The functions

$$\begin{aligned}
w_1 &= \phi_1 + \mathrm{i}\psi_1, \quad y < \eta(x),\\
w_2 &= \phi_2 + \mathrm{i}\psi_2, \quad y > \eta(x),
\end{aligned}\tag{16}$$

must be analytic in their respective domains. Following Forbes [13] and Hocking [8], and applying Cauchy's Theorem to w_j, $j = 1, 2$, on both regions, we obtain

$$\pi w_j(z_0) = \int_{\Gamma_j} \frac{w_j(z)}{z - z_0}\,\mathrm{d}z, \quad j = 1, 2 \tag{17}$$

where Γ_j, $j = 1, 2$ are the contours shown in Fig. 2, and z_0 lies on the boundary in each case at the point P. Now since, $w_j \to 0$, $j = 1, 2$ as $|z| \to \infty$, the contribution of that part of w_j, $j = 1, 2$ that consists of the circular arc can be shown to be zero. Thus we only need to integrate along the interface. Using an arclength variable, s, along the interface starting from the sink, then

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}\eta}{\mathrm{d}s}\right)^2 = 1, \tag{18}$$

and using the chain rule we can write

$$\begin{aligned}
\pi i w_1(z(s)) &= \int_{-\infty}^{\infty} \frac{w_1(z(t))\mathrm{d}z/\mathrm{d}t}{z(t) - z(s)}\mathrm{d}t,\\
-\pi i w_2(z(s)) &= \int_{-\infty}^{\infty} \frac{w_2(z(t))\mathrm{d}z/\mathrm{d}t}{z(t) - z(s)}\mathrm{d}t,
\end{aligned}\tag{19}$$

where s and t are both arclengths, but s defines a particular location and t is the variable of integration. These integrals must be interpreted in the Cauchy-principal-value sense.

Since ψ_1, ψ_2 are known along the interface from (15), the Eqs. 19 represent integral equations for ϕ_1 and ϕ_2, respectively. Taking the real parts and utilizing the symmetry of the situation about the line $x = 0$, i.e.,

$$\begin{gathered}
x(-s) = -x(s), \quad y(-s) = y(s), \quad x'(-s) = x'(s), \quad y'(-s) = -y'(s),\\
\phi_j(-s) = \phi_j(s), \quad \psi_j(-s) = -\psi_j(s), \quad j = 1, 2,
\end{gathered}$$

we may convert the integral equations into

$$\begin{aligned}
\phi_j(s) = \frac{\beta_j}{\pi}\int_0^{\infty} &\phi_j(t)\left(\frac{y'(t)\Delta x - x'(t)\Delta y}{\Delta x^2 + \Delta y^2} + \frac{y'(t)\Delta x_+ - x'(t)\Delta y}{\Delta x_+^2 + \Delta y^2}\right)\\
&+ \psi_j(t)\left(\frac{x'(t)\Delta x + y'(t)\Delta y}{\Delta x^2 + \Delta y^2} + \frac{x'(t)\Delta x_+ + y'(t)\Delta y}{\Delta x_+^2 + \Delta y^2}\right)\mathrm{d}t, \quad j = 1, 2,
\end{aligned}\tag{20}$$

where $\Delta x = x(t) - x(s)$, $\Delta x_+ = x(t) + x(s)$, $\Delta y = y(t) - y(s)$, and $\beta_1 = 1$, $\beta_2 = -1$.

The problem to be solved is the combination of the two integral equations given by (20) and the interface condition (8). No analytic solution exists for this highly nonlinear problem and therefore it must be solved numerically. The logarithmic singularity near the sink has to be treated carefully, but the following method was successful:

1. For the nonlinear integral equations (20), the domain $[0, \infty)$ of the independent variable s was truncated at a finite point, s_T, and the interval was discretized into the set of points s_j, $j = 1, 2, 3, \ldots, N$, where N is the number of points on the interface, and $s_0 = 0$, $s_N = s_T$. The distribution of these points was usually uniform in arclength s, but in some cases a quadratic distribution was used to crowd many points close to the region of greatest change near to the sink.

2. An initial guess was made for the unknown values of the correction terms of the velocity potentials, $\phi_1(s)$ and $\phi_2(s)$, and the derivative of the interface location $\eta'(s)$. The entry angle of the interface into the sink, α, and the non-dimensional flow rate, G, were assigned. If either of these parameters was left as an unknown, the method failed to converge.
3. The other variables, $x(s)$ and $y(s)$, were then computed by finding $x'(s)$ from (18) and then using numerical integration. A trapezoidal scheme was found to be adequate in all cases.
4. Using $x, \eta, x'(s), y'(s), \phi_1(s)$ and $\phi_2(s)$ along the interface, the error in (20) was computed and a damped Newton iteration scheme was applied to update the original guess.
5. Once ϕ_1, ϕ_2 were obtained, a forward difference scheme was used to calculate their derivatives and the error in the interface condition (9) was evaluated. If the error was small at all points on the interface, say less than 10^{-9}, the algorithm was stopped. Otherwise, Newton's method was used to update $\eta'(s)$, ϕ_1, ϕ_2, and repeat from step 3.

The accuracy of the numerical integration is crucial to the solution of the full problem. The singular part of the principal-value integral in (20) was removed by noting that

$$\int_0^{z_T} \frac{w_j(z)}{z-z_0}\mathrm{d}z = \int_0^{z_T} \frac{w_j(z)-w_j(z_0)}{z-z_0}\mathrm{d}z + w_j(z_0)\log\left(\frac{z_T-z_0}{z_0}\right), \tag{21}$$

where $z_N = z_T$ corresponds to the point at which the integral is truncated. It is also essential to include an approximation to the portion of the integral that is neglected. Both ϕ and ψ can be shown to behave like $O(s^{-1})$ as $s \to \infty$, so a simple correction term can be added to each integral to account for the truncation. The iteration scheme converged in only 4 or 5 iterations and solutions to graphical accuracy were found with N as small as $N = 80$, but most solutions were computed with $N = 200$, i.e., with 200 collocation points on the interface (which means 600 equations and unknowns).

3 Results and discussion

A series of computations was performed using the boundary-integral method, and the interface locations at different non-dimensional supercritical withdrawal parameter, G, and interface entry angle α, were obtained. It was found that there was a range of values of G for which solutions existed for each entry angle α.

The value of the viscosity ratio was kept at $\gamma = 1$ for all simulations, but behaviour for other values can be inferred from (12). Some solutions are shown in Fig. 3a. When the entry angle is not zero, the interface levels off, but at an infinite elevation according to the asymptote determined from equations (12) and (14), i.e.,

$$\eta \to \frac{-8\gamma G\alpha}{\pi(1+\gamma)+2\alpha(1-\gamma)}\log s, \tag{22}$$

where s is the arclength measured from the sink. This provides a good test of the numerical scheme and this asymptote is seen as a dashed line in Fig. 3a which shows interface profiles for $\alpha = -\pi/4$ and $G = 0.12, 0.72$ and 1.44. Clearly the numerical scheme is working well and has the correct behaviour for large x.

The results show that for each entry angle of the interface into the line sink there is a range of supercritical solutions that depend on the flow rate. It is found that as the withdrawal rate decreases, the interface near to (but not at) the sink steepens until it becomes vertical, at which point the method fails. This is clear in Fig. 3b which shows a close-up of (a).

As the magnitude of the entry angle increased (see Fig. 4) the range of G values for which solutions exist decreased, and when α began to approach $-\pi/2$ the numerical scheme struggled to converge, leading to the somewhat jagged appearance of the left edge of the domain in Fig. 4. However, those cases which did converge are highly accurate and repeatable, and were calculated with $N = 600$ points on the interface using a quadratic distribution of points so that many more were crowded near to the sink. As the entry angle approaches vertical it is clear that the solution range narrows to a small region about $G = 0.3$. It is worth noting that if $\alpha = 0$ there exist solutions

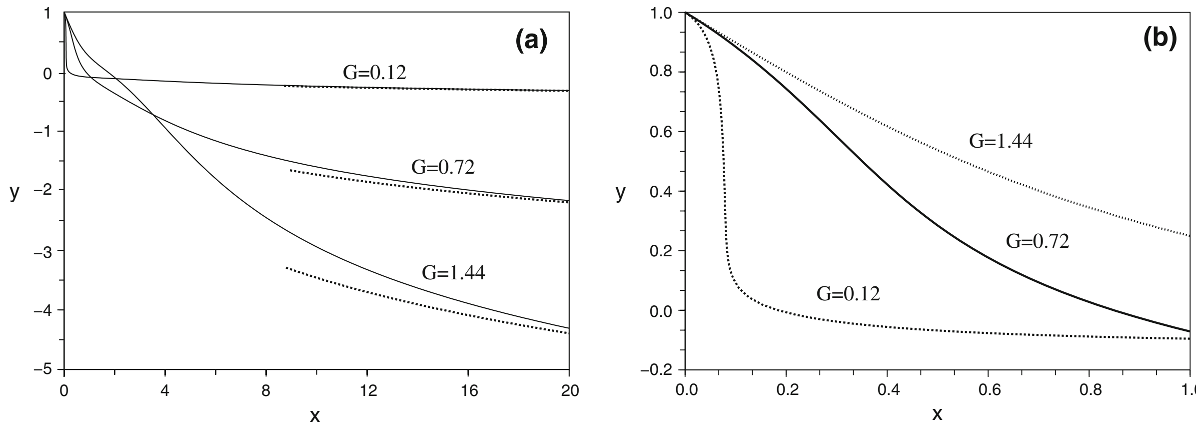


Fig. 3 **a** Interface shapes with $\alpha = -\pi/4$, for $G = 0.12$, 0.72 and 1.44 compared with the asymptotic solution given by (22). **b** Close-up view of the same showing behaviour near to the sink. Note the steepening of the surface as the non-dimensional flow rate, G, decreases

for any value of G because an horizontal interface, $\eta = 1$ is an exact solution. Unfortunately, it was not possible to compute solutions right up to $\alpha = -\frac{\pi}{2}$, and so we were unable to determine if this range narrows to a single point.

There is a known limiting solution computed by Bear and Dagan [2] using the Hodograph method for the single-layer flow with $\alpha = -\pi/2$, and this is shown as a '+' in Fig. 4. Bear and Dagan [2] use the distance from the sink to the cusp of the interface (which does not exist in the current work) as the length scale, but a recalculation of their G value for comparison gives the value $G_{cr} = \pi^{-1}$.

Finally, it is of interest to consider if there are any solutions where the interface levels off at a finite elevation that is different from the sink elevation. Such a branch of solutions was found and was investigated separately. It was shown earlier that this can only happen if the fluxes from within the two layers match. Figure 5 shows typical solutions and it is clear that as G decreases, just as in the other cases, the interface near the sink steepens until it becomes almost vertical at which point the numerical method fails. The magnitude of the steepest angle of the interface increases rapidly and gets very close to vertical when $G = 0.12$. Solutions of this type, with $\alpha = 0$, exist for all values of G greater than this minimum. Figure 6 shows the maximum value of the interface slope, $\eta'(x)$, just before it enters the sink for a series of decreasing values of G. It is clear that the slope steepens dramatically toward $-\frac{\pi}{2}$ as G decreases. This provides the limiting solution for this case.

4 Conclusions

The supercritical withdrawal or coning flow of two fluids of different density into a line sink in an homogeneous, isotropic two-dimensional aquifer was investigated. A coupled boundary-integral method was used to compute the interface shapes for the supercritical, coning case in which both fluids are drawn directly into the sink.

If we restrict attention to the case in which the interface levels off at finite elevation, i.e., $\alpha = 0$, we obtain solutions for all withdrawal rates above some minimum $G = 0.12$. As the value of G decreases toward this minimum the interface close to (but not at) the sink, steepens until it becomes vertical. In this case there is no maximum withdrawal rate, G, with the interface shape simply becoming flatter as G increases.

Allowing a non-zero entry angle into the sink means that the surface asymptotes to an infinite elevation, but the behaviour of the interacting parameters is the same, i.e., at each value of entry angle there is a range of solutions for differing values of G, and there is a minimum G value at which the interface near to the sink steepens to become close to vertical. There is also a maximum G for each angle beyond which no solutions were found. Thus as the

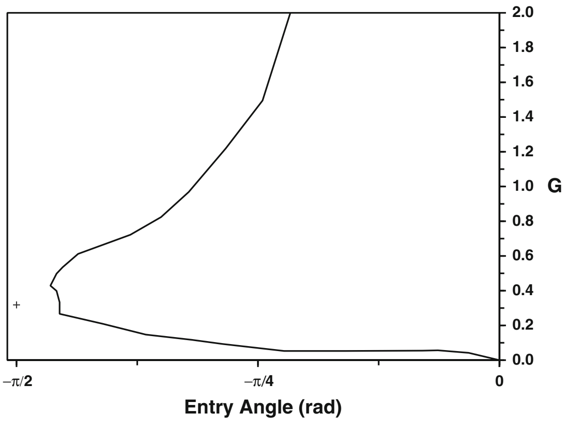


Fig. 4 Solution domain showing angle of entry against G. Solutions exist everywhere to the right of the given curve. The lower limit is characterised by the interface slope becoming very steep near to the sink, as in Fig. 3b, $G = 0.12$. The + is the solution of [2] for critical single-layer flow

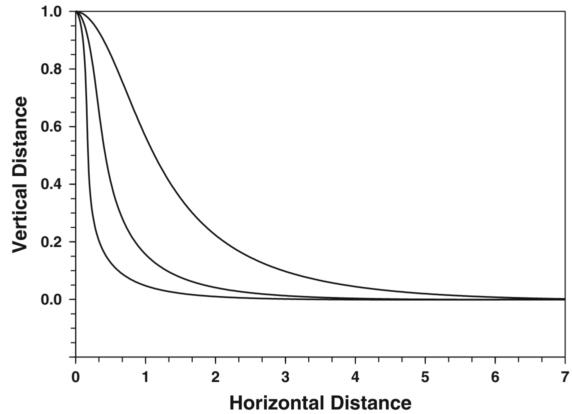
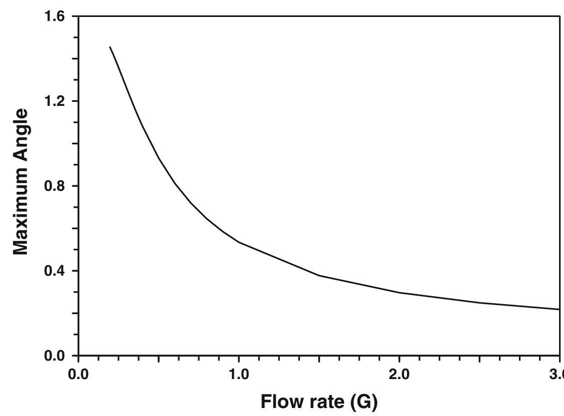


Fig. 5 Interface shapes for the special case with $\alpha = 0$ and $G = 0.16$, 0.24 and 0.81 (from left to right, respectively) for the unbounded two-layer flow where the interface levels off at $y = 0$

Fig. 6 Maximum slope on the interface near to (but not at) the sink as a function of flow rate for the special solution branch where entry angle into the line sink was zero, $\alpha = 0$. As G decreases the steepest angle approaches vertical as G approaches 0.12



magnitude of the entry angle, α, increases, the range of G values decreases, seeming to close in on a narrow range of values close to $G = 0.3$.

In the analogous surface water withdrawal problem, [8], the conclusions are clear. As the withdrawal rate decreases, the angle of entry of the interface into the sink increases in magnitude until it reaches vertical. This corresponds very closely to the limiting, steady, single-layer flow. Unfortunately, the conclusions that one may draw from the current work are not so clear.

The results from the two-fluid simulations show that for each value of entry angle, α, there is a range of G values that admit a solution. Therefore, Fig. 4 represents perhaps the main result of this work. As the entry angle approaches $\alpha = -\frac{\pi}{2}$ the solutions fall within an approximate range of $0.25 < G < 0.4$. The critical value may lie within this range or it may correspond to some other minimum value for a different α, in which the transition might involve a slight jump from two-fluid coning flow to a single-fluid flow as G decreases.

Whatever the situation, it seems almost certain that the critical transition occurs somewhere between $G = 0.25$ and $G = 0.4$, since all of the minimum G values for two-fluid flow fall within this range. This range of values compares exceptionally well with the limiting single-layer flow of Bear and Dagan [2] of $G = \pi^{-1}$. It is apparent that further research is required, and it is likely that both the stability and the evolution of the interface over time may be pivotal in determining which of the above steady-state solutions, if any, will evolve.

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